SNAP Centre Workshop

Logarithms

Exponential and Logarithmic Form

Logarithms offer a way of reorganizing the information given in an exponential expression, and the properties they exhibit are a direct result of this.

At its most basic, an equation in exponential form consists of three pieces of information: the **base** (a), which is the term being multiplied repeatedly, the **exponent** (b), which represents the number of times the multiplication is being repeated, and the **product** (c), which is the result of the repeated multiplication process.

Exponential Form $a^b = c$

An equation in logarithmic form uses the same three pieces of information, however, we refer to the product (c)as the *argument*.

Logarithmic Form $\log_a c = b$

Conceptually, logarithms pose the question "To what exponent do I need to raise my base in order to produce a given argument?"

Note: *In the following examples, the $a, b,$ and c labels given in the introduction are used to maintain consistency.*

Example 1 $2^6 = 64$ $\quad$ *Express in logarithmic form.*

Verbally, the equation given reads: "2 to the power of 6 equals 64"

Looking at the left side of the equation, we identify the base to be 2 and the exponent to be 6. On the right side, the product is 64, which will end up becoming the argument in logarithmic form.

$$a = 2 \qquad b = 6 \qquad c = 64$$

Once the information is gathered, expressing it in logarithmic form is simply a matter of rearrangement.

$$\mathbf{\log_2 64 = 6}$$

Rearranging our equation in logarithmic form provides the same information, but presents it in a different way.

Verbally, the equation reads: "Log-base-2 of 64 is equal to 6."

Conceptually, the left side poses the question, "To what exponent do we need to raise our base, 2, to get our product, 64?", and the right side provides the answer, "6."

Example 2 $\log_{\frac{3}{2}} \frac{27}{8} = 3$ $\quad$ *Express in exponential form.*

*As in **Example 1**, our goal here is to extract information from the equation, and rearrange it; the fractions do not change the overall process.*

From the left side, we get a base of $\frac{3}{2}$ and an argument of $\frac{27}{8}$, which will become our product in exponential form. From the right side, we find our exponent to be 3.

$$a = \frac{3}{2} \qquad b = 3 \qquad c = \frac{27}{8}$$

Once our information is gathered, all we have left to do in order to express the equation in exponential form is rearrange it.

$$\boldsymbol{\left(\frac{3}{2}\right)^{3} = \frac{27}{8}}$$

***Note:** Make sure the equation expressed in exponential form follows the correct order of operations. Had we not included the parentheses, the left side of the equation would have been $\frac{3^3}{2}$, which is not equal to the right side. Recalculating the right side is not an option since the value $\frac{27}{8}$ has been given by the equation in logarithmic form.*

Solving Logarithmic Equations and Evaluating Logarithmic Expressions

Logarithmic expressions are not always given in terms of constants; it is often the case that a logarithmic expression will contain a variable whose value can be solved for algebraically.

Example 3 $\log_3 t = 4$ $\qquad$ $Solve\ for\ t.$

Our first step in solving for t will be isolating it. This involves expressing the equation given to us in logarithmic form in exponential form.

Looking at the left side of the equation, we identify the base to be 3*, and the argument (which will become the product in exponential form) to be t. On the right side, we see that the exponent is* 4.

$$a = 3 \qquad b = 4 \qquad c = t$$

Expressed in exponential form, the equation becomes:

$$3^4 = t$$

Solving for t is now a matter of computing a straightforward exponential expression.

$$\boldsymbol{t = 3^4 = 71}$$

Evaluating a given logarithmic expression makes use of the same processes used to rearrange equations in logarithmic form.

Example 4 $\log_{10} 1000$ $\qquad$ $Evaluate\ the\ logarithm.$

This logarithmic expression can be evaluated most readily by first equating it to a variable of our choosing, then rearranging to express the equation in exponential form.

$$\log_{10} 1000 = p$$

As in the previous examples, we identify our base, argument, and exponent, then rearrange.

$$10^p = 1000$$

Next, we want to bring both terms to a common base. 10 *is a good choice, given the left side of our equation.* 1000*, by observation, is equal to* 10^3*.*

$$10^p = 10^3$$

Since both sides of our equation now consist of a common base raised to an exponent, we know the exponents must be equal, which gives us a value for p*.*

$$p = 3$$

Substituting p *back into our original equation gives us the value of the logarithmic expression in question.*

$$\mathbf{\log_{10} 1000 = 3}$$

Example 4 introduces us to $\log_{10}$ ("log-base-10"), which is also known as the **common logarithm**. Whenever we see "log" written without a specified base, we assume it is a common log and has an implied base of 10.

Similarly, $\ln$ is known as the **natural logarithm**, and denotes a logarithm with an implied base of e.

Adding and Subtracting Logarithms

Similar to the product rule of exponents where $a^x a^y = a^{x+y}$, logarithmic expressions possess an additive property whereby the sum of logarithms with the same base is equal to the logarithm (again, with the same base) of the product of our arguments. Expressed as an equation:

$$\log_a x + \log_a y = \log_a xy$$

Likewise, the difference of logarithms with the same base is equal to the logarithm of the quotient of arguments:

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

Note: The argument of the term being subtracted becomes the denominator in the equivalent expression.

Example 5 $\quad \log_6 18 + \log_6 2 \quad$ $Evaluate.$

Since our two terms have the same base, we can make use of the additive property.

$$= \log_6(18 \times 2)$$

$$= \log_6(36)$$

*As was the case in **Example 5**, our next step involves equating the expression to a variable of our choosing, and rearranging to exponential form.*

$$\log_6(36) = w$$

$$6^w = 36$$

Next, we identify 6 *as the ideal choice for a common base, and solve for* w*, which gives us our fully evaluated expression.*

$$6^w = 6^2$$

$$w = 2$$

$$\mathbf{\log_6 18 + \log_6 2 = 2}$$

Example 6 $\log 150 - \log 3 + \log 20$ $\quad Evaluate.$

Since we are not given a base explicitly, we assume a base of 10*. Since all terms share the same base, we can use the additive and subtractive properties of logarithms, making sure the arguments of terms being added become part of the equivalent expression's numerator, and arguments of terms being subtracted become part of the denominator.*

$$= \log\left(\frac{150\times 20}{3}\right)$$

$$= \log 1000$$

Instead of formally solving this expression, we can observe that our base is 10 *and our argument is* 1000*.* 10 *raised to the power* 3 *gives us* 1000*, which gives us our fully-evaluated expression.*

$$\mathbf{\log 150 - \log 3 + \log 20 = 3}$$

Note: Two of the most common mistakes with regards to logarithmic operations are encountered when adding and multiplying logarithmic expressions. To avoid making these mistakes yourself, keep the following in mind:

The sum of two logarithmic expressions is not equal to the logarithm of the sum of arguments:

$$\log x + \log y \neq \log(x + y)$$

Nor is the product of two logarithmic expressions equal to the logarithm of the product of arguments:

$$(\log x)(\log y) \neq \log(xy)$$

Arguments Raised to an Exponent

When a logarithmic expression's argument is a term raised to an exponent, the logarithmic expression can be manipulated so that the exponent is removed from the term in the argument, and the value of the exponent becomes a coefficient by which the entire logarithmic expression is multiplied.

$$\log_a(x^m) = m\log_a x$$

Example 7 $\log_5(125^6)$ $\quad Evaluate.$

Instead of evaluating the given argument of 125^6*, we can begin by removing the exponent from the argument and multiplying the entire term by that same value.*

$$= 6(\log_5 125)$$

Next, we can observe that our base of 5 *raised to the power* 3 *is equal to our argument of* 125*, giving us a value for our term in parentheses, and allowing us to fully evaluate the original expression.*

$$= 6(3)$$

$$= 18$$

$$\mathbf{\log_5(125^6) = 18}$$

Note: It is a common mistake to make the statement $(\log_a x)^m = m \log_a x$, however, this is incorrect. This rule is only applicable when the argument is raised to an exponent, not the entire logarithmic expression.

Change of Base Formula

The **change of base formula** is used in cases where a logarithmic expression with an uncommon base needs to be evaluated. It allows us to convert the expression into a fraction involving standard logarithmic bases.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The change of base formula is particularly helpful when using a calculator is an available option. log and ln are common calculator functions, and can be used to approximately evaluate logarithmic expressions with uncommon bases.

Example 9 $\log_7 100$ $Evaluate.$

There is no readily apparent way to evaluate this expression, so our first step is rearranging it according to the change of base formula, which will allow us to calculate an approximate value. Since most calculators have a ln function, we will use e as our new base.

$$= \frac{\ln 100}{\ln 7}$$

After rearranging, we can use our calculator to evaluate our numerator and denominator.

$$= \frac{4.605}{1.946}$$

$$= \mathbf{2.367}$$

Although we know our value is an approximation, it is still a good idea to check to see if it is a viable answer. Raising 7 *to the exponent* 2.367 *gives a value of* 100.08*, which is quite close to the argument in our original equation.*

Notable Logarithmic Expressions

$$\log_a a = 1$$

$$\log 10 = 1$$

$$\ln e = 1$$

Whenever a logarithm's base and argument are the same, the exponent is 1. This makes sense when examining the relationship between logarithmic and exponential expressions; any base value raised to the exponent 1 is going to be equal to the base value.

Common and natural logarithms are no exception. Since the bases 10 and e are implied, the arguments 10 and e will result in an exponent of 1.

$$\log_a 1 = 0$$

Regardless of a logarithm's base, whenever we are presented with the argument 1, the resulting exponent is 0. We know this because any value raised to the exponent 0 will be equal to 1.

$$\log_b b^n = n$$

If the argument of a logarithm is the logarithm's base raised to an exponent, the resulting exponent will be equal to the exponent in the argument. Thinking about it conceptually, this logarithm poses the question "b raised to what power will be equal to b^n?" The answer is n.

$$a^{\log_a b} = b$$

Anytime we raise a base to an exponent that is a logarithmic expression with that same base, we get the argument of our logarithmic expression as an answer.